

L^p-BOUNDEDNESS OF FLAG KERNELS ON HOMOGENEOUS GROUPS

P. GLOWACKI

ABSTRACT. We prove that the flag kernel singular integral operators of Nagel-Ricci-Stein on a homogeneous group are bounded on L^p , $1 < p < \infty$. The gradation associated with the kernels is the natural gradation of the underlying Lie algebra. Our main tools are the Littlewood-Paley theory and a symbolic calculus combined in the spirit of Duoandikoetxea and Rubio de Francia.

1. INTRODUCTION

Flag kernels on homogeneous groups have been introduced by Nagel-Ricci-Stein [8] in their study of quadratic CR -manifolds. They can be regarded as a generalization of Calderón-Zygmund singular kernels with singularities extending over the whole of the hyperspace $x_1 = 0$, where x_1 is the top level variable. The definition is complex (see below), as it involves cancellation conditions for each variable separately. However, the description of flag kernels in terms of their Fourier transforms is much simpler and bears a striking resemblance to that of the symbols of convolution operators considered independently by the author (in, e.g. [5]).

In Nagel-Ricci-Stein [8] we find an L^p -boundedness theorem for the very special flag kernels where the associated gradation consists of commuting subalgebras of the underlying Lie algebra of the homogeneous group. The natural question of what happens if the gradation is the natural gradation of the homogeneous Lie algebra is left open. The aim of this paper is to answer the question in the affirmative. We prove that such flag kernels give rise to bounded operators.

The smooth symbolic calculus mentioned above has been adapted to an extended class of flag kernels of small (positive and negative) orders and combined with a variant of the Littlewood-Paley theory built on a stable semigroup of measures with smooth densities very similar to the Poisson kernel on the Euclidean space. The strong maximal function of Christ [1] is also instrumental. The approach has been inspired by the well-known paper by Duoandikoetxea and Rubio de Francia [2]. The dependence of the present paper on Duoandikoetxea and Rubio de Francia [2] is evident throughout.

The class of flag kernels dealt with here is in fact an algebra. For this the reader is referred to [6] where also the L^2 -boundedness of flag kernels is proved solely by means of the symbolic calculus.

After this paper had been completed, a preprint of Nagel-Ricci-Stein-Wainger *Singular integrals with flag kernels on homogeneous groups I*, has been made available, where the L^p -boundedness theorem for flag kernels is proved. This comprehensive treatment of flag kernels on homogeneous groups has been announced for some time. Professor Stein has lectured a couple of times on the subject, see, e.g. [9]. The authors also use a version of Littlewood-Paley theory but otherwise the approach differs from the one presented here in many respects, the most important being our use of the symbolic calculus and

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partitions of unity related to a stable semigroup of measures. That is why we believe that what is presented here has an independent value and may count as a contribution to the theory.

2. PRELIMINARIES

Let \mathfrak{g} be a nilpotent Lie algebra with a fixed Euclidean structure and \mathfrak{g}^* its dual. Let $\delta_t x = tx$, $t > 0$ be a family of dilations on \mathfrak{g} and let

$$\mathfrak{g}_j = \{x \in \mathfrak{g} : \delta_t x = t^{p_j} \cdot x\}, \quad 1 \leq j \leq d,$$

where $1 = p_1 < p_2 < \dots < p_d$. Denote by

$$Q_j = p_j \cdot \dim \mathfrak{g}_j$$

the homogenous dimension of \mathfrak{g}_j . The homogeneous dimension of \mathfrak{g} is

$$Q = \sum_{j=1}^d Q_j.$$

We have

$$(2.1) \quad \mathfrak{g} = \bigoplus_{j=1}^d \mathfrak{g}_j, \quad \mathfrak{g}^* = \bigoplus_{j=1}^d \mathfrak{g}_j^*$$

and

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \begin{cases} \mathfrak{g}_k, & \text{if } p_i + p_j = p_k, \\ \{0\}, & \text{if } p_i + p_j \notin \mathcal{P}, \end{cases}$$

where $\mathcal{P} = \{p_j : 1 \leq j \leq d\}$.

Let

$$x \rightarrow |x| \approx \sum_{j=1}^d \|x_j\|^{1/p_j}$$

be a homogeneous norm on \mathfrak{g} smooth away from the origin. Let also

$$|x|_j = |(x_1, x_2, \dots, x_j, 0, \dots, 0)|, \quad 1 \leq j \leq d.$$

In particular, $|x|_1 = |x_1|$, and $|x|_d = |x|$. Another notation will be applied to \mathfrak{g}^* . For $\xi \in \mathfrak{g}^*$,

$$|\xi|_j = |(0, \dots, 0, \xi_j, \xi_{j+1}, \dots, \xi_d)|, \quad 1 \leq j \leq d.$$

In particular, $|\xi|_1 = |\xi|$, and $|\xi|_d = |\xi_d|$.

We shall also regard \mathfrak{g} as a Lie group with the Campbell-Hausdorff multiplication

$$xy = x + y + r(x, y),$$

where $r(x, y)$ is the (finite) sum of terms of order at least 2 in the Campbell-Hausdorff series for \mathfrak{g} . Under this identification the homogeneous ideals

$$\mathfrak{g}^{(k)} = \bigoplus_{j=k}^d \mathfrak{g}_j$$

are normal subgroups.

In expressions like D^α or x^α we shall use multiindices

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d),$$

where

$$\alpha_k = (\alpha_{k1}, \alpha_{k2}, \dots, \alpha_{kn_k}), \quad n_k = \dim \mathfrak{g}_k = \dim \mathfrak{g}_k^*,$$

are themselves multiindices with positive integer entries corresponding to the spaces \mathfrak{g}_k or \mathfrak{g}_k^* . The homogeneous length of α is defined by

$$|\alpha| = \sum_{k=1}^d |\alpha_k|, \quad |\alpha_k| = p_k(\alpha_{k1} + \alpha_{k2} + \cdots + \alpha_{kn_k}).$$

The Schwartz space of smooth functions which vanish rapidly at infinity along with their derivatives will be denoted by $\mathcal{S}(\mathfrak{g})$. For a tempered distribution K , that is a continuous linear functional on $\mathcal{S}(\mathfrak{g})$, we shall write

$$\langle K, f \rangle = \int_{\mathfrak{g}} f(x) K(x) dx, \quad f \in \mathcal{S}(\mathfrak{g}),$$

without implying thereby that K is a locally integrable function.

Even though the flag kernels are our prime concern here we need a broader class of kernels to properly deal with them. In [7], we proposed a natural generalization of the flag kernels of Nagel-Ricci-Stein. Let

$$\|f\|_{(k)} = \max_{|\alpha| \leq Q_k+1} \sup_{x \in \mathfrak{g}_k} (1 + |x|)^{Q_k+1} |D^\alpha f(x)|$$

be a fixed norm in the Schwartz space $\mathcal{S}(\mathfrak{g}_k)$. Let

$$\mathcal{N} = \{\nu = (\nu_1, \nu_2, \dots, \nu_d) : |\nu_k| < Q_k, 1 \leq k \leq d\}.$$

Let $\nu \in \mathcal{N}$. We define the class $\mathcal{F}(\nu)$ by induction on the homogeneous step d . When $d = 0$ the elements of $\mathcal{F}(\emptyset)$ are simply constants. If $d \geq 1$, we say that a distribution $K \in \mathcal{S}^*(\mathfrak{g})$ is in $\mathcal{F}(\nu)$ if it is smooth away from the hyperspace $x_1 = 0$ and satisfies the following conditions:

i) For every multiindex α ,

$$(2.2) \quad |D^\alpha K(x)| \leq C_\alpha |x_1|^{-\nu_1 - Q_1 - |\alpha_1|} |x_2|^{-\nu_2 - Q_2 - |\alpha_2|} \cdots |x_d|^{-\nu_d - Q_d - |\alpha_d|}$$

for $x_1 \neq 0$;

ii) For any $1 \leq k \leq d$,

$$(2.3) \quad \langle K_{R,\varphi}, f \rangle = R^{-\nu_k} \int_{\mathfrak{g}} \varphi(Rx_k) f(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_d) K(x) dx$$

is in $\mathcal{F}(\nu_{(k)})$ on $\oplus_{j \neq k} \mathfrak{g}_j$, where $\nu_{(k)} = (\nu_1, \dots, \nu_{k-1}, \nu_{k+1}, \dots, \nu_d)$, and this is uniform in $\varphi \in \mathcal{S}(\mathfrak{g}_1)$ with $\|\varphi\|_{(k)} \leq 1$ and $R > 0$. (Note that the meaning of *uniform boundedness* of a family of members of $\mathcal{F}(\nu)$ is obvious in the case $d = 0$ and, for $d \geq 1$, can be defined by induction.)

For every N , we define a norm $\|\cdot\|_{\nu,N}$ in $\mathcal{F}(\nu)$ as the maximum of all the bounds occurring in the definition. First, we let

$$s_N^\nu(P) = \max_{|\alpha| \leq N} \sup_{x_1 \neq 0} \prod_{k=1}^d |x_k|^{Q_k + \nu_k + |\alpha_k|} |D^\alpha K(x)|.$$

and, if $d = 1$,

$$\|K\|_{\nu_1,N} = s_N^{\nu_1}(K) + \sup_{|\varphi\|_{(1)} \leq 1} \sup_{R > 0} R^{-\nu_1} \langle K, \varphi \circ \delta_R \rangle.$$

If $d > 1$, we let

$$\|K\|_{\nu,N} = s_N^\nu(K) + \max_{1 \leq k \leq d} \sup_{\|\varphi\|_{(k)} \leq 1} \sup_{R > 0} \|K_{R,\varphi}\|_{\nu_{(k)},N}.$$

Thus, $\mathcal{F}(\nu)$ can be regarded as a locally convex topological vector space. Let us remark that $\mathcal{F}(0) = \mathcal{F}(0, 0, \dots, 0)$ is exactly the class of flag kernels of Nagel-Ricci-Stein [8] (see Corollary 3.7 of [7]).

For a $K \in \mathcal{S}^*(\mathfrak{g})$, let

$$\langle \tilde{K}, f \rangle = \int_{\mathfrak{g}} f(x^{-1})K(dx), \quad f \in \mathcal{S}(\mathfrak{g}).$$

The following three propositions have been proved in [6] and [7].

2.4. Proposition ((Theorem 2.5 of [6])). *Let $K \in \mathcal{F}(0)$ be a flag kernel on \mathfrak{g} . The convolution operator $f \rightarrow f \star \tilde{K}$ defined initially on $\mathcal{S}(\mathfrak{g})$ extends uniquely to a bounded operator on $L^2(\mathfrak{g})$.*

2.5. Proposition ((Proposition 1.5 of [7])). *Let $\nu \in \mathcal{N}$. A distribution K is in $\mathcal{F}(\nu)$ if and only if its Fourier transform is locally integrable, smooth for $\xi_d \neq 0$, and satisfies*

$$(2.6) \quad |D^\alpha \hat{K}(\xi)| \leq C_\alpha |\xi|_1^{\nu_1 - |\alpha_1|} \dots |\xi|_d^{\nu_d - |\alpha_d|}, \quad \xi_d \neq 0.$$

Cf. also the original Theorem 2.3.9 of Nagel-Ricci-Stein [8] for kernels $K \in \mathcal{F}(0)$.

2.7. Proposition (Theorem 4.8 of [7]). *Let $\nu, \mu, \nu + \mu \in \mathcal{N}$. Let $K \in \mathcal{F}(\nu)$, $L \in \mathcal{F}(\mu)$. Let $\varphi = \otimes_{k=1}^d \varphi_k \in C_c^\infty(\mathfrak{g})$ be equal to 1 in a neighbourhood of 0. There exists a $P = P_{K,L} \in \mathcal{F}(\nu + \mu)$ such that*

$$P = \lim_{\epsilon \rightarrow 0} K_\epsilon \star L$$

in the sense of distributions, where

$$\langle K_\epsilon, f \rangle = \int_{\mathfrak{g}} \varphi(\epsilon x) f(x) K(dx), \quad f \in \mathcal{S}(\mathfrak{g}).$$

Moreover, the mapping $(K, L) \rightarrow P_{K,L}$ is continuous.

3. SEMIGROUPS OF MEASURES

Following Folland-Stein [3], we say that a function φ belongs to the class $\mathcal{R}(a)$, where $a > 0$, if it is smooth and

$$(3.1) \quad |D^\alpha \varphi(x)| \leq C_\alpha (1 + |x|)^{-Q-a-|\alpha|}, \quad \text{all } \alpha.$$

3.2. Proposition. *Let $\varphi \in \mathcal{R}(a)$ for some $0 < a < 1$ and let $\int \varphi = 0$. Then $\varphi \in \mathcal{F}(a)$.*

Proof. The size condition (2.2) follows by (3.1). To verify the cancellation condition (2.3) let $f \in \mathcal{S}(\mathfrak{g})$ and $R > 0$. Then

$$\begin{aligned} \int_{\mathfrak{g}} f(Rx) \varphi(x) dx &= \int_{\mathfrak{g}} (f(Rx) - f(0)) \varphi(x) dx \\ &\leq \int_{|x| \leq R^{-1}} (f(Rx) - f(0)) \varphi(x) dx + \int_{|x| \geq R^{-1}} (f(Rx) - f(0)) \varphi(x) dx \\ &\leq \|f\| \left(R \int_{|x| \leq R^{-1}} |x|^{-Q-a+1} dx + 2 \int_{|x| \geq R^{-1}} |x|^{-Q-a} dx \right) \\ &\leq CR^a \|f\|, \end{aligned}$$

where $\|\cdot\|$ is a Schwartz class norm. □

Let

$$\langle P, f \rangle = \lim_{\epsilon \rightarrow 0} \int_{|x| \geq \epsilon} (f(0) - f(x)) \frac{dx}{|x|^{Q+1}}, \quad f \in \mathcal{S}(\mathfrak{g}).$$

The distribution P is an infinitesimal generator of a continuous semigroup of probability measures with smooth densities

$$h_t(x) = t^{-Q} h(t^{-1}x),$$

where $h \in \mathcal{R}(1)$ and $P^N h \in \mathcal{R}(N)$ for $N = 1, 2, \dots$. In other words,

$$h_t \star h_s = h_{t+s}, \quad t, s > 0.$$

and

$$\frac{d}{dt} \Big|_{t=0} \langle h_t, f \rangle = - \langle P, f \rangle, \quad f \in \mathcal{S}(\mathfrak{g}),$$

The operator $Pf = f \star P$ is essentially selfadjoint with $\mathcal{S}(\mathfrak{g})$ for its core domain. The reader is referred to [4] for proofs and details.

For $0 < a < 1$

$$(3.3) \quad \langle P^a, f \rangle = \frac{1}{\Gamma(-a)} \int_0^\infty t^{-1-a} \langle \delta_0 - h_t, f \rangle dt = \frac{1}{\Gamma(1-a)} \int_0^\infty t^{-a} \langle Ph_t, f \rangle dt$$

defines a homogeneous distribution smooth away from the origin (cf., e.g. Yosida [10]).

3.4. Proposition. *For every $0 < a < 1$,*

$$P^a h \in \mathcal{R}(a) \quad \text{and} \quad \int_{\mathfrak{g}} P^a h(x) dx = 0.$$

Proof. By (3.3),

$$P^a h(x) = \frac{1}{\Gamma(1-a)} \int_0^\infty t^{-a} Ph_{t+1}(x) dt,$$

whence

$$\begin{aligned} |D^\alpha P^a h(x)| &\leq \frac{C_\alpha}{\Gamma(1-a)} \int_0^\infty \frac{t^{-a} dt}{(t+1+|x|)^{Q+1+|\alpha|}} \\ &\leq C'_\alpha \int_0^\infty \frac{t^{-a} dt}{(\frac{t}{1+|x|} + 1)^{Q+1+|\alpha|}} \cdot (1+|x|)^{-Q-1-|\alpha|} \\ &\leq C''_\alpha \int_0^\infty \frac{t^{-a} dt}{(t+1)^{Q+1+|\alpha|}} \cdot (1+|x|)^{-Q-a-|\alpha|}, \end{aligned}$$

as required.

Now, for every $t > 0$,

$$\int h_t dx = 1.$$

Therefore,

$$\int Ph_t dx = -\frac{d}{dt} \int h_t dx = 0, \quad t > 0.$$

which combined with (3.3) gives the second part of the assertion. \square

4. LITTLEWOOD-PALEY THEORY

From now on we fix the function $\varphi = P^{1/2}h_{1/2}$.

4.1. *Remark.* By the results of the previous section, φ is a smooth function satisfying the estimates

$$(4.2) \quad |D^\alpha \varphi(x)| \leq C_\alpha (1 + |x|)^{-Q-1/2-|\alpha|}.$$

Moreover, $\varphi \in \mathcal{F}(1/2)$.

4.3. **Lemma.** *We have*

$$f = \int_0^\infty f \star \varphi_t \star \varphi_t \frac{dt}{t}, \quad f \in \mathcal{S}(\mathfrak{g}).$$

Proof. By the semigroup properties,

$$-\frac{d}{dt} f \star h_t = f \star P h_t = \frac{1}{t} f \star (\varphi_t \star \varphi_t),$$

whence

$$\int_\epsilon^M f \star \varphi_t \star \varphi_t \frac{dt}{t} = f \star h_\epsilon - f \star h_M.$$

Now, if $\epsilon \rightarrow 0$ and $M \rightarrow \infty$, the expression on the right hand side tends to f in the sense of distributions. \square

Let $T = (t_1, \dots, t_d) \in \mathbf{R}_+^d$. We shall regard \mathbf{R}_+^d as a product of copies of the multiplicative group \mathbf{R}^+ . We shall write

$$T^a = (t_1^a, \dots, t_d^a), \quad TS = (t_1 s_1, \dots, t_d s_d), \quad \frac{dT}{T} = \frac{dt_1 \dots dt_d}{t_1 \dots t_d}, \quad a \in \mathbf{R}.$$

Let φ_k be the counterpart of φ for \mathfrak{g} replaced by $\mathfrak{g}^{(k)}$, $1 \leq k \leq d$. Let

$$\Phi_k = \delta_k \otimes \varphi_k,$$

where δ_k stands for the Dirac delta at $0 \in \oplus_{j=1}^{k-1} \mathfrak{g}_j$. Let

$$\Phi = \Phi_1 \star \Phi_2 \star \dots \star \Phi_d,$$

and

$$\Phi_T = (\Phi_1)_{t_1} \star \dots \star (\Phi_d)_{t_d}, \quad T \in \mathbf{R}_+^d.$$

4.4. **Corollary.** *We have*

$$\Phi \in |\mathcal{F}|(1/2) := \bigcap_{\epsilon \in \{-1, 1\}} \mathcal{F}(\epsilon_1/2, \dots, \epsilon_d/2).$$

Furthermore,

$$f = \int_{\mathbf{R}_+^d} f \star \Phi_T \star \widetilde{\Phi}_T \frac{dT}{T}, \quad f \in \mathcal{S}(\mathfrak{g}).$$

Proof. By Remark 4.1,

$$\Phi_k \in \mathcal{F}(0, \dots, 0, 1/2, 0, \dots, 0) \cap \mathcal{F}(0, \dots, 0, -1/2, 0, \dots, 0),$$

where the only nonzero term stands on the k -th position. Therefore the first part of our assertion follows by Proposition 2.7. The second one is a consequence of Lemma 4.3. \square

4.5. Proposition. *The Paley-Littlewood square function*

$$G_\Phi(f)(x) = \left(\int_{\mathbf{R}_+^d} |f \star \Phi_T(x)|^2 \frac{dT}{T} \right)^{1/2},$$

is bounded as an operator on $L^p(\mathfrak{g})$. In other words, for every $1 < p < \infty$, there is a constant $C_{\varphi,p} > 0$ such that

$$\|G_\Phi(f)\|_p \leq C_{\varphi,p} \|f\|_p, \quad f \in \mathcal{S}(\mathfrak{g}).$$

Proof. The proof is implicitly contained in Folland-Stein [3] (see Theorem 6.20.b and Theorem 7.7) so we dispense ourselves with presenting all details.

We start with defining some Hilbert spaces and operators. Let $X_0 = \mathbf{C}$ and

$$X_k = L^2(\mathbf{R}_+^k, \frac{dT}{T}), \quad 1 \leq k \leq d.$$

For a given $x \in \mathfrak{g}$, let $F_k(x) : X_{k-1} \rightarrow X_k$ be given by

$$F_k(x)m(t_1, \dots, t_{k-1}, t_k) = (\varphi_k)_{t_k}(x_k, \dots, x_d)m(t_1, \dots, t_{k-1}). \quad m \in X_{k-1}.$$

Finally, let $W_k : C_c(\mathfrak{g}, X_{k-1}) \rightarrow C_0(\mathfrak{g}, X_k)$ be the operator

$$W_k f(x)(T, t_k) = (f \star F_k)(x)(T, t_k) = \int_{\mathfrak{g}^{(k)}} (\varphi_k)_{t_k}(y) f(xy)(T) dy,$$

where $T = (t_1, \dots, t_{k-1})$. Note that W_k acts only on (x_k, \dots, x_d) -variable.

We claim that

$$W_k : L^2(\mathfrak{g}, X_{k-1}) \rightarrow L^2(\mathfrak{g}, X_k)$$

is an isometry. In fact, by definition of Φ_k ,

$$\begin{aligned} \|W_k f\|_{L^2(\mathfrak{g}, X_k)}^2 &= \int_{\mathfrak{g}} \|W_k f(x)\|_{X_k}^2 dx \\ &= \int_{\mathfrak{g}} dx \int_0^\infty \frac{dt}{t} \int_{\mathbf{R}_+^{k-1}} \frac{dT}{T} \int_{\mathfrak{g}^{(k)}} |(\varphi_k)_t(y) f(xy)(T)|^2 dy \\ &= \int_{\mathbf{R}_+^{k-1}} \frac{dT}{T} \int_0^\infty \frac{dt}{t} \int_{\mathfrak{g}} \int_{\mathfrak{g}^{(k)}} |(\varphi_k)_t(y) f(xy)(T)|^2 dy dx, \\ &= \int_{\mathbf{R}_+^{k-1}} \frac{dT}{T} \int_0^\infty \frac{dt}{t} < f_T \star (\Phi_k)_t, f_T \star (\Phi_k)_t > = \|f\|_{L^2(\mathfrak{g}, X_{k-1})}^2, \end{aligned}$$

where $f_T(x) = f(x)(T)$.

Another property of W_k that is needed is the following. For every α

$$(4.6) \quad \|D^\alpha F_k(x)\|_{(X_{k-1}, X_k)} \leq C_\alpha |x|_k^{-Q-|\alpha|}.$$

This follows readily from (4.2) specialized to φ_k :

$$|D^\alpha \varphi_k(x)| \leq C_\alpha (1 + |x|_k)^{-Q-1/2-|\alpha|}.$$

As a bounded operator from $L^2(\mathfrak{g}, X_{k-1})$ to $L^2(\mathfrak{g}, X_k)$ satisfying (4.6) is W_k a vector-valued kernel of type 0, and, by Theorem 6.20.b of Folland-Stein [3], maps $L^p(\mathfrak{g}, X_{k-1})$ into $L^p(\mathfrak{g}, X_k)$ boundedly for every $1 < p < \infty$.

This implies our assertion. In fact,

$$G_\Phi(f)(x) = \|f \star F_1 \star \dots \star F_d(x)\|_{X_d},$$

and therefore

$$\|G_\Phi(f)\|_{L^p(\mathfrak{g})} = \|T_d T_{d-1} \dots T_1 f\|_{L^p(\mathfrak{g}, X_d)} \leq C \|f\|_{L^p(\mathfrak{g}, X_0)} = C \|f\|_p.$$

□

A word of comment on the symbol Φ_T would be appropriate here. The notation may suggest that the functions Φ_T are dilates of a single function. They are not, but they have estimates of this form, which is our justification. The same applies to the symbol K_T below. In the next section we are going to use the same notation for the “real” dilates of a function. We hope the reader will not get confused.

5. THE STRONG MAXIMAL FUNCTION

For a function F on \mathfrak{g} and a $T \in \mathbf{R}_+^d$, let

$$F_T(x) = F_{(t_1, t_2, \dots, t_d)}(x) = t_1^{-Q_1} t_2^{-Q_2} \dots t_d^{-Q_d} F(t_1 x_1, t_2 x_2, \dots, t_d x_d).$$

The strong maximal function on \mathfrak{g} is defined by

$$\mathbf{M}f(x) = \sup_{T \in \mathbf{R}_+^d} \int_{|y| \leq 1} |f(x(Ty)^{-1})| dy = \sup_T |f \star (\chi_B)_T(x)|,$$

where χ_B stands for the characteristic function of the unit ball $B = \{x \in \mathfrak{g} : |x| \leq 1\}$, and $Ty = (t_1 y_1, \dots, t_d y_d)$. A theorem of Michael Christ asserts that for every $1 < p < \infty$ there exists a constant $C > 0$ such that

$$\|\mathbf{M}f\|_p \leq C \|f\|_p, \quad f \in L^p(\mathfrak{g}),$$

that is, \mathbf{M} is of (p, p) type (see Christ [1]).

We shall need the following corollary to the Christ theorem. Let

$$\gamma(t) = \min\{t, t^{-1}\}, \quad t > 0.$$

5.1. Corollary. *Let*

$$F(x) = \prod_{j=1}^d \gamma(|x_j|)^a |x_j|^{-Q_j}, \quad x \neq 0,$$

for some $a > 0$. Then the maximal function

$$M_F f(x) = \sup_{T \in \mathbf{R}_+^d} |f \star F_T(x)|$$

is of (p, p) type for $1 < p < \infty$.

Proof. Let B_j be the unit ball in \mathfrak{g}_j and let $|B_j|$ be the Lebesgue measure of B_j . Let $D = B_1 \times \dots \times B_d$. Then for every simple positive function $h \leq F$ of the form

$$h(x) = \sum_R c_R \chi_D(R^{-1}x), \quad R = (r_1, r_2, \dots, r_d) \in \mathbf{R}_+^d,$$

we have

$$h_T(x) = \sum_R c_R r_1^{Q_1} r_2^{Q_2} \dots r_d^{Q_d} (\chi_D)_{RT}(x) = \frac{C \|h\|_1}{|D|} (\chi_D)_{RT}(x),$$

and therefore

$$M_F f(x) \leq \frac{C \|F\|_1}{|D|} \mathbf{M}f(x),$$

which completes the proof. □

6. FLAG KERNELS

We keep the notation established in previous sections.

6.1. Lemma. *Let*

$$K_{T,S} = \widetilde{\Phi_{TS}} \star K \star \Phi_T, \quad T, S \in \mathbf{R}_+^d.$$

Then $K_{T,S} \in \mathcal{F}(0)$ uniformly, and satisfy the estimates

$$(6.2) \quad |D^\alpha \widehat{K}_{T,S}(\xi)| \leq C_\alpha \gamma(S)^{1/2} |\xi|_1^{-|\alpha_1|} \dots |\xi|_d^{-|\alpha_d|},$$

where

$$\gamma(S) = \gamma(s_1) \gamma(s_2) \dots \gamma(s_d).$$

Proof. By the first part of Corollary 4.4, $\Phi_T \in |\mathcal{F}|(1/2)$ with bounds uniformly proportional to $\gamma(T)^{1/2}$. Note that

$$\gamma(TS) \leq \gamma(T) \cdot \gamma(S).$$

Thus, our assertion follows by Proposition 2.7. \square

We let

$$K_T = K \star \Phi_T, \quad T \in \mathbf{R}_+^d.$$

6.3. Lemma. *For every T , K_T is an integrable function, and the maximal operator*

$$(6.4) \quad K_\Phi^* f(x) = \sup_T |f \star \widetilde{K}_T|(x)|$$

is of type (p, p) for all $1 < p < \infty$.

Proof. Observe that by Proposition 2.4, $K_T \in L^2(\mathfrak{g})$ so it is a function. Moreover, by Corollary 4.4 and Proposition 2.7, it is a smooth away from $x_1 = 0$, and satisfies

$$|K_T(x)| \leq C \gamma(T)^{1/2} \gamma(|x|_1)^{1/2} |x|_1^{-Q_1} \dots \gamma(|x|_d)^{1/2} |x|_d^{-Q_d}$$

uniformly in T so that $K_T \leq C F_T$, where F_T is a dilate of

$$F(x) = \gamma(|x|_1)^{1/2} |x|_1^{-Q_1} \dots \gamma(|x|_d)^{1/2} |x|_d^{-Q_d}.$$

This shows that K_T is integrable. The second part of our claim follows by Corollary 5.1 and the above. \square

We turn to the main result of this paper. The reader may wish to compare the proof we give with that of Theorem B and the preceding lemma of Duoandicoetxea-Rubio de Francia [2].

6.5. Theorem. *Let K be a flag kernel on \mathfrak{g} . Then the singular integral operator*

$$f \rightarrow f \star \widetilde{K}, \quad f \in \mathcal{S}(\mathfrak{g}),$$

extends uniquely to a bounded operator on $L^p(\mathfrak{g})$ for all $1 < p < \infty$.

Proof. Let $f, h \in \mathcal{S}(\mathfrak{g})$. We have

$$\begin{aligned} \langle f \star \widetilde{K}, h \rangle &= \int_{\mathbf{R}_+^d} \frac{dS}{S} \int_{\mathbf{R}_+^d} \frac{dT}{T} \langle f \star \Phi_T, h \star \Phi_{TS} \star \widetilde{\Phi_{TS}} \star K \star \Phi_T \rangle \\ &= \int_{\mathbf{R}_+^d} \frac{dS}{S} \int_{\mathbf{R}_+^d} \frac{dT}{T} \langle f_T, h_{TS} \star K_{T,S} \rangle, \end{aligned}$$

where

$$f_T = f \star \Phi_T, \quad h_{TS} = h \star \Phi_{TS}, \quad K_{T,S} = \widetilde{\Phi_{TS}} \star K \star \Phi_T.$$

We are going to estimate

$$\langle L_S f, h \rangle = \int_{\mathbf{R}_+^d} \frac{dT}{T} \langle f_T, h_{TS} \star K_{T,S} \rangle$$

for a given S . Let us start with L^2 -estimates. We have

$$|\langle L_S f, h \rangle| \leq \left(\int_{\mathfrak{g}} \int_{\mathbf{R}_+^d} |f_T(x)|^2 \frac{dT}{T} dx \right)^{1/2} \cdot \left(\int_{\mathfrak{g}} \int_{\mathbf{R}_+^d} |h_{TS} \star K_{T,S}(x)|^2 \frac{dT}{T} dx \right)^{1/2}.$$

By (6.2) and Proposition 2.4, the operators $f \rightarrow f \star K_{T,S}$ are bounded with norm estimates uniformly proportional to $\gamma(S)^{1/2}$ so that, by Proposition 4.5,

$$\begin{aligned} |\langle L_S f, h \rangle| &\leq C \gamma(S)^{1/2} \|G_\Phi(f)\|_2 \|G_\Phi(h)\|_2 \\ &\leq C_1 \gamma(S)^{1/2} \|f\|_2 \|h\|_2, \end{aligned}$$

that is,

$$(6.6) \quad \|L_S f\|_2 \leq C_1 \gamma(S)^{1/2} \|f\|_2, \quad f \in \mathcal{S}(\mathfrak{g}).$$

For $1 < p < 2$ and $f, h \in \mathcal{S}(\mathfrak{g})$,

$$\begin{aligned} |\langle L_S f, h \rangle| &\leq \int_{\mathfrak{g}} \left(\int_{\mathbf{R}_+^d} |f_T(x)|^2 \frac{dT}{T} \right)^{1/2} \left(\int_{\mathbf{R}_+^d} |h_{TS} \star K_{T,S}(x)|^2 \frac{dT}{T} \right)^{1/2} dx \\ &\leq C_1 \|G_\Phi(f)\|_p \left(\int_{\mathfrak{g}} \left(\int_{\mathbf{R}_+^d} |h_{TS} \star K_{T,S}(x)|^2 \frac{dT}{T} \right)^{q/2} dx \right)^{1/q} \\ &= C_2 \|f\|_p \cdot \left\| \int_{\mathbf{R}_+^d} |h_{TS} \star K_{T,S}(\cdot)|^2 \frac{dT}{T} \right\|_{q/2}^{1/2}, \end{aligned}$$

where $1/p + 1/q = 1$. Note that $q > 2$. Thus, there exists a nonnegative function u with $\|u\|_r = 1$, where $2/q + 1/r = 1$, such that

$$\left\| \int_{\mathbf{R}_+^d} |h_{TS} \star K_{T,S}(\cdot)|^2 \frac{dT}{T} \right\|_{q/2} = \int_{\mathfrak{g}} \int_{\mathbf{R}_+^d} |h_{TS} \star K_{T,S}(x)|^2 \frac{dT}{T} \cdot u(x) dx.$$

Now,

$$\begin{aligned} h_{TS} \star K_{T,S} &= (h \star \Phi_{TS}) \star (\widetilde{\Phi_{TS}} \star K \star \Phi_T) \\ &= (h \star \Phi_{TS} \star \widetilde{\Phi_{TS}}) \star (K \star \Phi_T) = h'_{TS} \star K_T. \end{aligned}$$

Recall also that, by Lemma 6.3, K_T are integrable functions. Therefore, by Lemma 6.3 again,

$$\begin{aligned} \left\| \int_{\mathbf{R}_+^d} |h'_{TS} \star K_T(\cdot)|^2 \frac{dT}{T} \right\|_{q/2} &\leq C_1 \int_{\mathfrak{g}} \int_{\mathbf{R}_+^d} |h'_{TS}|^2 \star |K_T|(x) \cdot u(x) dx \frac{dT}{T} \\ &\leq C_2 \int_{\mathfrak{g}} \int_{\mathbf{R}^d} |h'_{TS}(x)|^2 \frac{dT}{T} \cdot K_\Phi^* u(x) dx \\ (6.7) \quad &\leq C_3 \|G_\Phi(h)\|_q^2 \cdot \|K_\Phi^* u\|_r \leq C_4 \|h\|_q^2, \end{aligned}$$

where we have used the estimate

$$\begin{aligned} |h'_{TS} \star K_T(x)|^2 &\leq \left(\int_{\mathfrak{g}} |h'_{TS}(xy^{-1})| \cdot |K_T(y)|^{1/2} \cdot |K_T(y)|^{1/2} dy \right)^2 \\ &\leq \int_{\mathfrak{g}} |h'_{TS}|^2(xy^{-1}) \cdot |K_T|(y) dy \cdot \int_{\mathfrak{g}} |K_T(y)| dy \\ &\leq C |h'_{TS}|^2 \star |K_T|(x), \end{aligned}$$

the integrals

$$\int_{\mathfrak{g}} |K_T(x)| dx \leq C$$

being uniformly bounded, as can be seen from the proof of Lemma 6.3. Therefore,

$$(6.8) \quad \|L_S f\|_p \leq C_1 \|f\|_p.$$

Now, by interpolating between (6.6) and (6.8), we get

$$\|L_S f\|_p \leq C_2 \gamma(S)^{\epsilon_p} \|f\|_p,$$

where $\epsilon_p > 0$ depends only on p , and, finally,

$$\|f \star \tilde{K}\|_p \leq C_3 \left(\int_{\mathbf{R}_+^d} \gamma(S)^{\epsilon_p} \frac{dS}{S} \right) \cdot \|f\|_p = C_4 \|f\|_p,$$

which proves our case for $1 < p \leq 2$. The result for $2 < p < \infty$ follows by duality. \square

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INSTITUTE OF MATHEMATICS, UNIVERSITY OF WROCLAW, PL. GRUNWALDZKI 2/4, 50-384 WROCLAW,
POLAND